A Fast Track to Relativity

In this paper we derive all of the results of the Special Theory of Relativity (STR) using a novel approach. This approach, which we call "Fast Track to STR" is concise and exact. It starts with time dilation and Doppler effect, then derives the addition of velocities formula and introcuces the "half speed" in STR. Equipped with that we derive the STR formulations of conservation of mass and conservation of momentum, which in turn allow us to calculate the relativistic expression of kinetic energy. All this is done on the first 6 pages.
In sections 7 to 19 we draw some important conclusions, show some old and a new proofs of the most popular formula of physics and give alternative ways to derive the basic results of STR.
In section 20 we introduce, 'post festum', the Lorentz transformations in order to derive the seldom used general formulas for the addition of velocities, aberration and Doppler shift. The last sections pay homage to Newton and give a short discussion of the logical background of STR.

Several links to the online edition of my book "Epstein Explains Einstein" (EEE for short) lead to specific illustrations and examples. What is missing are the relevant transformations of the electric and magnetic field. For this consult e.g.
https://www.relativity.li/en/maxwell2/max_00_en or
https://www.physastromath.ch/uploads/myPdfs/Relativ/STR with Four-Vectors.pdf

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Martin Gubler alias David Eckstein
gub@stafag.ch

1. Time Dilation and Doppler Shift

In the realm of acoustics two particular situations are distinguished:

a) The sender is at rest in the wave-propagating medium, the receiver is moving away with speed v from the sender. The corresponding Doppler-formula is

$$f_R = f_S \cdot \frac{c - v}{c} \tag{1.1}$$

b) The receiver is at rest in the wave-propagating medium, the sender is moving away with speed v from the receiver. The corresponding Doppler-formula is

$$f_R = f_S \cdot \frac{c}{c + v} \tag{1.2}$$

In STR, the basic assumptions are that the speed of light c is the same in all inertial frames, and, furthermore, that this speed is independent of the movement of the sender. These basic assumptions are just what Maxwell's equations for electro-magnetic waves imply. Unlike sound waves there is no special coordinate system for light waves in which the medium for the wave propagation is at rest. There is no need for something like the 'Ether'.

Since there is no distinguished coordinate frame only relative speeds can be measured. Then both of the above situations should yield the same amount of Doppler shift. So something has to happen to frequencies if the sender and the receiver are moving relative to each other. If frequencies are influenced by movement *time itself must be influenced by movement* ! Time is the only quantity involved in the counting of the number of oscillations when the speed of wave propagation has a fixed value. This means that the flow of time does not have the same speed in coordinate frames moving relative to each other and we have to say goodbye to Newton's idea of absolute time !

Let us assume the existence of a function $r(v)$ depending on relative speed v so that the following equation holds

$$\Delta t_v = \Delta t_0 \cdot r(v)$$

Δt_v is a time interval measured in the moving system, and Δt_0 is the corresponding time interval measured at rest. $r(v)$ cannot equal 1 for $v \neq 0$, formulas (1.1) and (1.2) are different.

We do not impose any restrictions on the function $r(v)$. Let us assume $r(v)$ to be *smaller* than 1 for $v \neq 0$ (the text could also be formulated for the other case without changing the outcome).

In case a) the receiver is moving, so his watch runs *slower* by the factor $r(v)$. Therefore he will measure a *higher* frequency, in his longer seconds he will count *more* oscillations. (1.1) has to be adjusted to

$$f_R = f_S \cdot \frac{c - v}{c} \cdot \frac{1}{r(v)}$$

In case b) the receiver is at rest, and the clock of the fast-moving sender runs *slow*. Hence the frequency of the sender is diminuished and the receiver measures a *lower* frequency. (1.2) has to be adjusted to

$$f_R = f_S \cdot \frac{c}{c + v} \cdot r(v)$$

In STR both of the above formulas should yield the same amount of Doppler shift. Therefore we get

$$\frac{c - v}{c} \cdot \frac{1}{r(v)} = \frac{c}{c + v} \cdot r(v)$$

or

$$r(v)^2 = \frac{(c - v)}{c} \cdot \frac{(c + v)}{c} = \frac{c^2 - v^2}{c^2} = 1 - \frac{v^2}{c^2}$$

The factor $r(v)$ turns out to be

$$\boldsymbol{r(v)} = \sqrt{\mathbf{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}} \tag{1.3}$$

The negative solution might be interesting for a science-fiction-story ...

Replacing $r(v)$ by (1.3) we get the correct Doppler shift formula in this special situation of relative speed along the line of sight between receiver and emitter. The *longitudinal Doppler shift* is controlled by the formula

$$\boldsymbol{f_R} = f_S \cdot \frac{c}{c + v} \cdot r(v) = f_S \cdot \frac{c}{c + v} \cdot \sqrt{\frac{(c - v) \cdot (c + v)}{c^2}} = \boldsymbol{f_S} \cdot \sqrt{\frac{(\boldsymbol{c} - \boldsymbol{v})}{(\boldsymbol{c} + \boldsymbol{v})}} \tag{1.4}$$

$$\boldsymbol{f_R} = f_S \cdot \frac{c - v}{c} \cdot \frac{1}{r(v)} = f_S \cdot \frac{c - v}{c} \cdot \sqrt{\frac{c^2}{(c - v) \cdot (c + v)}} = \boldsymbol{f_S} \cdot \sqrt{\frac{(\boldsymbol{c} - \boldsymbol{v})}{(\boldsymbol{c} + \boldsymbol{v})}} \qquad (1.5$$

with v denoting the speed of *increasing* distance of sender and receiver. In both cases we come to the same result.

The principle of relativity, denying the existence of a unique ether system, yields immediately the formulas of time dilation and longitudinal Doppler shift.

Graphic representations of the three Doppler formulas and a beautiful application of the longitudinal Doppler shift are presented in https://www.relativity.li/en/epstein2/read/d0_en/d6_en . In that publication the sign of relative speed v is positive if sender and receiver are *approaching* each other and thus the plus and minus signs are reversed.

Now, consider a sender moving at right angle to the line of sight between sender and receiver. Actually there is no change of distance between sender and receiver, but the senders frequency as seen by the receiver is *lowered* because of time dilation. Therefore we have

$$\boldsymbol{f_R} = \boldsymbol{f_S} \cdot \sqrt{\mathbf{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}} \tag{1.6}$$

This effect is called *transversal Doppler shift*. It is a purely relativistic effect unknown in 'classical' physics. It is much harder to demonstrate the transversal Doppler shift by experiments than the longitudinal Doppler shift, because the effect depends on the square of the small number v/c and not on v/c itself.

Formulas for the general case of Doppler shift are presented in section 23.

2. Addition of Parallel Velocities

We can use the formula of longitudinal Doppler shift to derive the STR formula for the addition of parallel velocities. The idea comes from Hermann Bondi ("Relativity and Common Sense", 1962, new edition by Dover Publications 1980) and is also presented by David Mermin in "It's About Time" (Princeton University Press 2005).

Let B be moving in positive x_A-direction of A with velocity v as measured by A, and let C be moving in positive x_B-direction of B with velocity u as measured by B. As usual both x-directions should merge into one. Now let C emit radiation of frequency f_C directed to B and A. Following (1.4) B is receiving that radiation at a frequency of

$$f_B = \sqrt{\frac{c - u}{c + u}} \cdot f_C$$

The radiation passes B with frequency f_B and, somewhat later, reaches A who will measure the frequency

$$f_A = \sqrt{\frac{c - v}{c + v}} \cdot f_B = \sqrt{\frac{c - v}{c + v}} \cdot \sqrt{\frac{c - u}{c + u}} \cdot f_C$$

Let z be the yet unknown velocity of C as measured by A . (1.4) tells us

$$f_A = \sqrt{\frac{c - z}{c + z}} \cdot f_C$$

Comparing both terms for f_A and solving for z we get

$$\boldsymbol{z = \frac{v + u}{1 + \frac{v \cdot u}{c^2}}} \qquad (2.1)$$

If both v and u are small compared with c , then (2.1) hardly differs from $z = v + u$, the result of the 'classical' addition of parallel velocities following Galileo and Newton.
Inserting c for u or v (or for both of them !) results in $z = c$. Obviously, the basic assumptions of STR do not lead to self-contradictions !

3. 'Half the Speed' and 'Twice the Speed' in STR

Now let us ask for the speed w that, if added to itself according to (2.1), yields a given speed v :

$$v = \frac{w + w}{1 + \frac{w \cdot w}{c^2}}$$

We shall call w 'half the speed' of v in STR, and v is called 'twice the speed' of w. Solving for w we get

$$\boldsymbol{w = \frac{v}{1 + \sqrt{1 - \frac{v^2}{c^2}}}} \qquad (3.1)$$

If v is small compared with c the root term comes close to 1 and hence w comes close to v / 2 , the 'classical' answer. The root term is always smaller than 1 , so w is always somewhat greater as v / 2 . As v approaches the speed of light w approaches the value v !

Jerzy Kocik shows in the American Journal of Physics (Vol. 80, Nr. 8, p. 737f) how to add velocities in STR with ruler and compass. Triggered by his publication my friend Alfred Hepp and me have worked out a small paper promoting the usage of 'half the speed' in STR: https://www.physastromath.ch/uploads/myPdfs/Relativ/Relativ_06_en.pdf

We will use this idea of 'half the speed' in the next section to derive the STR terms for 'dynamic mass' and 'momentum'. Equation (3.1) will help to avoid some annoying algebra. And in section 8 we will derive a new result from (3.1): Multiplying the STR term of momentum by half the speed you get the STR term of kinetic energy !

Speed w creates a third point of view in all situations where we have two inertial frames S and S' with relative speed v . If the inertial frame T is moving with speed w relative to S, the constellation is completely symmetric as observed from the system T : System S is moving to the left with speed $-w$ while System S' is moving to the right with speed w .

4. The Perfectly Inelastic Collision

In coordinate frame S let two identical bodies move in a perfectly symmetric way towards each other. We allow their identical masses to depend on their velocities, but they do not have to do so. Their momenta are given by

$$m_w \cdot w \qquad \text{and} \qquad m_w \cdot (-w)$$

Total momentum is zero. So, after the perfectly inelastic collision, we have a single mass M_0 at rest in frame S .

Now let us observe this collision in a frame S' moving with speed $-w$ as seen from S . In S' the second body is at rest while the first moves with 'twice the speed' v . After the collision the single mass M_w moves with speed w in S'. Conservation of momentum and conservation of mass are **the core credos of physics**. The corresponding equations for the collision as observed in frame S' are

$$\text{I} \qquad m_v \cdot v \;=\; M_w \cdot w$$

$$\text{II} \qquad m_v + m_0 \;=\; M_w$$

Substituting M_w in the first equation by using the second and replacing w according to (3.1) we get

$$m_v \cdot v \;=\; (\, m_v + m_0 \,) \cdot w \;=\; (\, m_v + m_0 \,) \cdot \frac{v}{1+\sqrt{1-\frac{v^2}{c^2}}}$$

Dividing by v we find the definition of 'dynamic mass' :

$$\boldsymbol{m_v \;=\; \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}} \tag{4.1}$$

Hence, relativistic momentum is given by

$$\boldsymbol{p \;=\; m_v \cdot v \;=\; \frac{m_0 \cdot v}{\sqrt{1-\frac{v^2}{c^2}}}} \tag{4.2}$$

Only these definitions make equations I and II come true ! Conservation of mass only holds for 'dynamic masses', and the definition of momentum needs an additional relativistic twist.

$w = v/2$ would imply $m_v = m_0$ and $M_w = M_0 \;=\; 2 \cdot m_0$. The 'slight' correction (2.1) brings to the formula for the addition of velocities has a deep impact !

This derivation is presented by Max Born in his influential book "Die Relativitätstheorie Einsteins" (first edition 1920). The above presentation is much simpler thanks to the 'half speed' formula (3.1).
The content of sections 11 and 18 can be found in Born's book as well. In section 18 we will present his derivation of relativistic momentum that is independent of conservation of mass **and** of conservation of momentum !

5. Total Energy, Kinetic Energy and Rest Energy

We follow the standard path to calculate the relativistic expression for kinetic energy; that is we are going to calculate the work needed to accelerate a body from $v_0 = 0$ to the final velocity v_{end} . With

$$dE = F \cdot ds \qquad \text{and} \qquad F = \frac{dp}{dt} \qquad (\text{ Newton's second law })$$

we get

$$dE = \frac{dp}{dt} \cdot ds = \frac{dp}{dv} \cdot \frac{dv}{dt} \cdot ds = \frac{dp}{dv} \cdot \frac{ds}{dt} \cdot dv = \frac{dp}{dv} \cdot v \cdot dv$$

and hence

$$E_{kin} = \int_0^{v_{end}} \frac{dp}{dv} \cdot v \cdot dv$$

From formula (4.2) of the last section we find

$$\boldsymbol{\frac{dp}{dv} = m_0 \cdot \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}} \tag{5.1}$$

and, together with (4.1), the integral yields

$$\boldsymbol{E_{kin} = m_0 \cdot c^2 \cdot \left(\frac{1}{\sqrt{1 - \frac{{v_{end}}^2}{c^2}}} - 1\right) = m_{v_{end}} \cdot c^2 - m_0 \cdot c^2 = \Delta m \cdot c^2} \tag{5.2}$$

Performing work on m or supplying energy to m results in an increase of mass according to

$$\boldsymbol{\Delta W^{\checkmark} = \Delta E = \Delta m \cdot c^2} \tag{5.3}$$

Energy and mass can be converted into each other. The *rest mass* m_0 corresponds to the *rest energy* E_0 of amount $m_0 \cdot c^2$, and *total energy* is given by

$$\boldsymbol{E_{tot} = E_0 + E_{kin} = m_v \cdot c^2 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot c^2} \tag{5.4}$$

Conservation of energy and conservation of dynamic mass melt into the same theorem. By choice it can be formulated as conservation of energy or conservation of dynamic mass.

Nowadays there are many examples illustrating the conversion of mass into energy or vice versa. Compare the corresponding chapters in 'EEE':
https://www.relativity.li/en/epstein2/read/f0_en/f3_en
https://www.relativity.li/en/epstein2/read/f0_en/f4_en
https://www.relativity.li/en/epstein2/read/f0_en/f5_en

6. Total Energy, Momentum and the Pythagorean Theorem

Let us subtract the squared rest energy from the squared total energy:

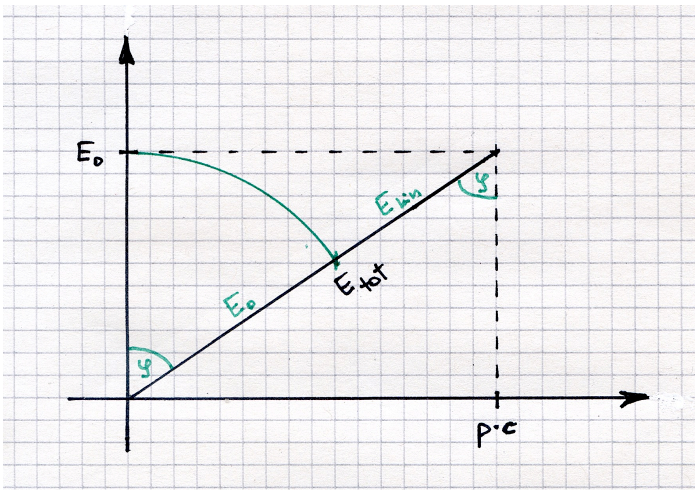
$$E_{tot}{}^2 - E_0{}^2 = \frac{m_0{}^2 \cdot c^4}{1 - \frac{v^2}{c^2}} - m_0{}^2 \cdot c^4 = m_0{}^2 \cdot c^4 \cdot \left(\frac{1}{1 - \frac{v^2}{c^2}} - 1 \right) =$$

$$= m_0{}^2 \cdot c^4 \cdot \left(\frac{1 - \left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2}} \right) = m_0{}^2 \cdot c^4 \cdot \left(\frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} \right) = \frac{m_0{}^2 \cdot v^2}{1 - \frac{v^2}{c^2}} \cdot c^2 = p^2 \cdot c^2$$

We get the amazing equation

$$\boldsymbol{E_0{}^2 + p^2 \cdot c^2 = E_{tot}{}^2} \tag{6.1}$$

Rest energy, momentum multiplied by the speed of light and total energy form the sides of a right triangle. In STR energy and momentum are similarly connected as time and space. If you are familiar with Epstein diagrams or with 4-vectors this is a simple fact. For more details see https://www.relativity.li/en/epstein2/read/e0_en/e4_en .



For the angle φ in this triangle we find

$$sin(\varphi) = \frac{p \cdot c}{E_{tot}} = \frac{m_v \cdot v \cdot c}{m_v \cdot c^2} = \frac{v}{c} \equiv \beta_v \tag{6.2}$$

and

$$cos(\varphi) = \frac{E_0}{E_{tot}} = \sqrt{1 - \frac{v^2}{c^2}} \equiv \frac{1}{\gamma_v} \tag{6.3}$$

These are the definitions of the traditional terms β and γ .

7. Total Energy, Momentum and the Full Speed

We may divide formula (5.4) by formula (4.2), or we can simply state

$$\frac{E_{tot}}{c^2} = m_v = \frac{p}{v}$$

Or we read from the picture in section 6

$$\frac{p \cdot c}{E_{tot}} = \sin\varphi = \frac{v}{c}$$

Total energy, momentum and speed v are connected by the equation

$$\boldsymbol{p \cdot c^2 = E_{tot} \cdot v} \qquad (7.1)$$

Formula (7.1) allows e.g. to calculate the speed of the center-of-mass system of some particles from total momentum and total energy of these particles.

8. Kinetic Energy, Momentum and 'Half the Speed'

Let us start with (6.1):

$$m_v{}^2 \cdot c^4 = m_0{}^2 \cdot c^4 + m_v{}^2 \cdot v^2 \cdot c^2$$

Dividing by c^2 and rearranging the terms we get

$$(m_v{}^2 - m_0{}^2) \cdot c^2 = m_v{}^2 \cdot v^2$$

Dividing by $(m_v + m_0)$ we get the expression (5.2) for kinetic energy on the left side:

$$(m_v - m_0) \cdot c^2 = \frac{m_v{}^2}{m_v + m_0} \cdot v^2 = \frac{m_v}{1 + \frac{m_0}{m_v}} \cdot v^2 = \frac{m_v}{1 + \sqrt{1 - \frac{v^2}{c^2}}} \cdot v^2 = m_v \cdot v \cdot \frac{v}{1 + \sqrt{1 - \frac{v^2}{c^2}}} = m_v \cdot v \cdot w$$

where w stands for 'half the speed' of v (see section 3). We get a pretty formula connecting kinetic energy, momentum and half the speed:

$$\boldsymbol{E_{kin} = m_v \cdot v \cdot w = p \cdot w} \qquad (8.1)$$

The same formula holds in 'classical' physics ! There we have

$$E_{kin} = \frac{1}{2} \cdot m \cdot v^2 = m \cdot v \cdot \frac{v}{2} = p \cdot w$$

It is far from being obvious that the relativistic expression for kinetic energy approaches the classical term if the velocities are small compared with the velocity of light. Obviously, the terms for momentum and half the speed do so, and hence does their product, too.

I am not aware of a text book presenting (8.1).

9. Momentum and Energy of Light Particles

(5.1) implies that no finite amount of energy can accelerate a body of some rest mass to the speed of light. Since light quanta or photons move with that speed, their rest mass has to be zero. Nevertheless they carry energy and momentum. Equation (6.1) says for a particle with $m_0 = 0$

$$0 + p^2 \cdot c^2 = {E_{tot}}^2$$

and hence

$$\boldsymbol{E = E_{tot} = E_{kin} = p \cdot c} \tag{9.1}$$

We can deduce (9.1) also by inserting the speed of light c for v in (7.1) or by inserting c for half the speed of c in (8.1).

Together with Planck's formula $E = h \cdot f$ we get the important relations

$$\boldsymbol{E = h \cdot f = p \cdot c} \tag{9.2}$$

and

$$\boldsymbol{p = \frac{E}{c} = \frac{h \cdot f}{c} = \frac{h}{\lambda}} \tag{9.3}$$

Radiation of frequency f consists of a stream of particles transmitting energy $h \cdot f$ and momentum $p = h \cdot f \,/\, c$. Taking this (at the time revolutionary) point of view Einstein explained in 1905 all the confusing phenomena of the photoelectric effect.

It was well known since 1884 that light carries momentum and energy. John Henry Poynting derived the details starting from Maxwell's equations, and he proved (9.1) to be true for electro-magnetic radiation.

A beautiful illustration of light pressure give the comets: Their tails always turn away from the sun. When the comet has passed perihelion and moves away from the sun its tail flies ahead of it ! The pressure of sun light blows the ions and dust particles away from sun. The following picture shows both the ion tail and the dust tail of comet Hale-Bopp. The heavier dust particels are harder to accelerate, and so the tail splits into different parts:

http://astronomy.swin.edu.au/sao/imagegallery/Hale-Bopp.jpg (1997)

10. $E = m \cdot c^2$ by Conservation of Momentum and $E = p \cdot c$

Figure a) shows a body of rest mass m_0 and two quanta of energy moving symmetrically towards that body. Each quantum carries momentum p and energy $E = p \cdot c$. After absorbing those quanta the body stays at rest due to symmetry or conservation of momentum. His energy is increased by $\Delta E = 2 \cdot E$, and his mass may be m_1 :

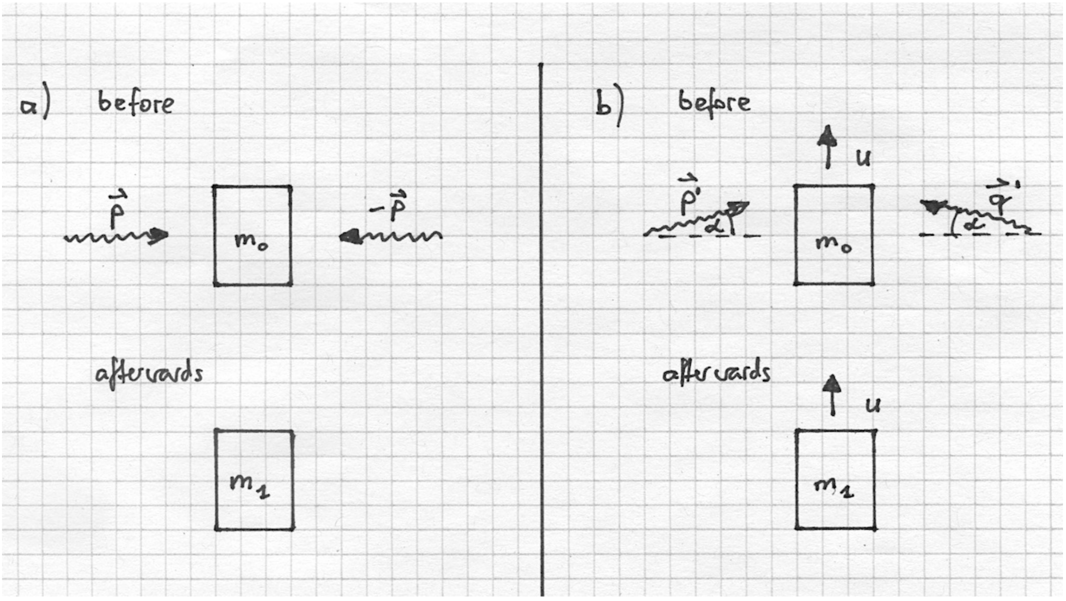


Figure b) shows the same process in a coordinate frame moving with speed u downwards. The bodies speed is u before and after the absorption. The quanta fall in by an angle $\propto$ to the horizontal line. They still move with speed c , and so we have $sin(\propto) = u/c$. The momenta p' and q' may slightly differ from p . Conservation of momentum in the direction of u means

$$\gamma_u \cdot m_1 \cdot u \; = \; \gamma_u \cdot m_0 \cdot u \; + \; 2 \cdot p' \cdot sin(\propto) \; = \; \gamma_u \cdot m_0 \cdot u \; + \; 2 \cdot p' \cdot \frac{u}{c}$$

Dividing by u (or by $\gamma_u \cdot u$) we have

$$m_1 \; = \; m_0 \; + \; 2 \cdot \frac{p'}{c} \cdot \frac{1}{\gamma_u}$$

That equation holds for all velocities u , u may be as small as you like. Hence the equation holds in the limit $u \to 0$, too! But if u approaches zero, p' approaches p (and γ_u approaches 1), and we get

$$m_1 - m_0 \; = \; 2 \cdot \frac{p}{c} \; = \; 2 \cdot \frac{E}{c^2} \; = \; \frac{\Delta E}{c^2}$$

or

$$\Delta m \; = \; \frac{\Delta E}{c^2} \qquad (10.1)$$

You might as well drop the terms printed in red for very small velocities u !

Einstein published this magnificent derivation in 1946. It is to be found in "Out of My Later Years", Random House 1993, section 14

11. $E = m \cdot c^2$ by Conservation of Momentum and $E = p \cdot c$

Figure a) shows two identical particles with rest mass m_0 moving towards each other with speeds v and $-v$. Total momentum is zero, so O, the center of mass, is at rest. Both particles have the same distance to O.

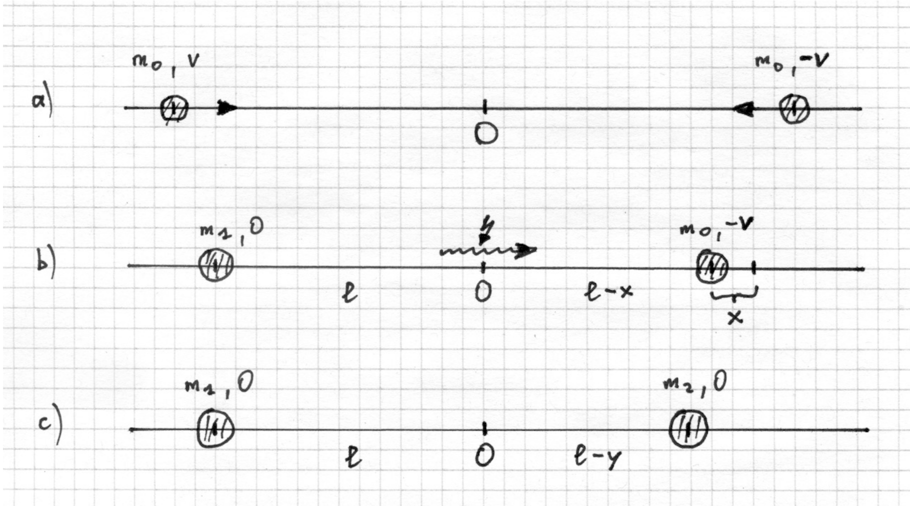


Figure b) shows the state of our system after the emission of a flash with energy E and momentum $p = \gamma \cdot m_0 \cdot v$ by the left particle. At time $\Delta t = l/c$ after that emission, the flash is in O. The left particle is at rest now, having the mass m_1 . At that time the right particle is in distance

$$l - x \;=\; l - \Delta t \cdot v \;=\; l - \frac{l}{c} \cdot v \;=\; l \cdot \left(1 - \frac{v}{c}\right)$$

to the center of mass O. Hence the equation of equilibrium is

$$l \cdot m_1 \;=\; \gamma \cdot m_0 \cdot (l - x) \;=\; \gamma \cdot m_0 \cdot l \cdot \left(1 - \frac{v}{c}\right)$$

The dynamic mass of the flash is of no influence to the balance because he is at O in that very moment.
We divide by l and use the equation $E = p \cdot c = \gamma \cdot m_0 \cdot v \cdot c$ to get

$$m_1 \;=\; \gamma \cdot m_0 \;-\; \gamma \cdot m_0 \cdot \frac{v}{c} \;=\; \gamma \cdot m_0 \;-\; \frac{p}{c} \;=\; \gamma \cdot m_0 \;-\; \frac{E}{c^2}$$

If the energy of a body is reduced by E the dynamic mass of that body is reduced by $\frac{E}{c^2}$!! (11.1)

In reversed time order that means that the supply of some additional energy E increases the dynamic mass by the amount $\frac{E}{c^2}$. Figure c) shows the state of our system after the absorption of the flash (with its momentum and energy) by the right particle. The particle is at rest now, and its mass is

$$m_2 \;=\; \gamma \cdot m_0 \;+\; \frac{E}{c^2}$$

Hence we get

$$\boldsymbol{m_2 - m_1} \;=\; \left(\gamma \cdot m_0 + \frac{E}{c^2}\right) - \left(\gamma \cdot m_0 - \frac{E}{c^2}\right) \;=\; 2 \cdot \frac{E}{c^2} \;=\; \frac{\boldsymbol{\Delta E}}{\boldsymbol{c^2}} \qquad (11.2)$$

In the beginning, both particles had the same rest mass and the same dynamic mass. After the transmission of energy and momentum the particles mass (both rest mass and dynamic mass ...) show a difference of $\Delta E / c^2$.

Energy and dynamic mass are convertible. The factor of conversion is the square of the speed of light:

$$\boldsymbol{\Delta E = \Delta m \cdot c^2} \qquad (11.3)$$

Dividing the equation for conservation of energy by c^2 we get the equation for conservation of dynamic mass. Rest mass is not conserved, our calculations show

$$m_1 + m_2 = \left(\gamma \cdot m_0 - \frac{E}{c^2}\right) + \left(\gamma \cdot m_0 + \frac{E}{c^2}\right) = 2 \cdot \gamma \cdot m_0 > 2 \cdot m_0$$

In two steps the kinetic energies of the particles are converted to additional rest mass in the above experiment.
Of course, total energy and total momentum are the same in all states of the experiment.

The basic idea to use the stability of the center of mass when particles within a closed system exchange energy and momentum comes from Einstein. He uses a box around the particles to force them to come to rest again, which needs some additional discussions, because there is no such thing as a rigid body in STR.
Francesco Cester finally got rid of that box in his book "Newton and Relativity" (Books on Demand, 2018). He works then with low-speed-approximations that are well justified if you have the exchanche of some photons in mind. But with that approximations the difference between rest mass and dynamic mass disappears.
The actual section is a enhanced version of Cester's presentation. It shows, by the way, that conservation of dynamic mass is a consequence of conservation of momentum.

You may omit all the γ-s in the above derivation, arguing that the velocity v is allowed to be as small as you like. Then you have another derivation of $\Delta E = \Delta m \cdot c^2$ that does not presuppose any relativistic findings about 'dynamic mass' and 'relativistic momentum'. However, you will miss then the important point that ΔE corresponds in general to an increase in *dynamic mass* and not in rest mass. An increase in rest mass can show up in special situations.

12. $E = m \cdot c^2$ by Conservation of Momentum and $E = h \cdot f$

A body at rest in system S with mass m_0 simultaneously emits in opposite directions two quanta of radiation (both with energy $h \cdot f$). Their momenta add up to zero, and so the body stays at rest in system S . The body may have rest mass m_1 after the emission.

Let be a system S' moving along the path of the emitted quanta with relative speed v as measured by S. The emitting body has speed v in System S' before and after the emission. Respecting the equation of conservation of momentum in S' and using (4.2), (9.3) and the Doppler shift formula (1.4) we get

$$\frac{m_0 \cdot v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_1 \cdot v}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{h \cdot f}{c} \cdot \sqrt{\frac{c+v}{c-v}} - \frac{h \cdot f}{c} \cdot \sqrt{\frac{c-v}{c+v}}$$

Rearranging this equation and doing some algebra we find

$$(m_0 - m_1) \cdot v = \frac{h \cdot f}{c} \cdot \sqrt{1 - \frac{v^2}{c^2}} \cdot \left(\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} - \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \right) =$$

$$= \frac{h \cdot f}{c} \cdot \sqrt{1 - \frac{v}{c}} \cdot \sqrt{1 + \frac{v}{c}} \cdot \left(\frac{\sqrt{1 + \frac{v}{c}}}{\sqrt{1 - \frac{v}{c}}} - \frac{\sqrt{1 - \frac{v}{c}}}{\sqrt{1 + \frac{v}{c}}} \right) =$$

$$= \frac{h \cdot f}{c} \cdot \left(\left(1 + \frac{v}{c}\right) - \left(1 - \frac{v}{c}\right) \right) = \frac{h \cdot f}{c} \cdot \frac{2v}{c}$$

Dividing by v we get

$$\boldsymbol{\Delta m = 2 \cdot h \cdot f \,/\, c^2 = \Delta E \,/\, c^2} \qquad (12.1)$$

Radiating away the energy ΔE reduces the rest mass of the body by $\Delta E \,/\, c^2$.

This calculation was stimulated by the book "Newton and Relativity" by Francesco Cester (Books on Demand, 2018).

Cester himself refers to an article of Fritz Rohrlich in the American Journal of Physics (Nr. 58 of April 1990). Rohrlich does the calculation by using the acoustic Doppler formula (1.1) instead of (1.4). This is justified by the fact that v may be as small as you like. Because of this approximation Rohrlich draws the wrong conclusion that the emitted energy is the same in both frames of reference.

However, the relativistic calculation shows that $\Delta E' = \Delta E \cdot \gamma$. It's exactly this result Einstein uses in his late 1905 paper entiteld "Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?". We will present his line of arguments in the next section. He uses conservation of energy instead of conservation of momentum.

13. $E = m \cdot c^2$ by Conservation of Energy and $E = h \cdot f$

In september 1905 Einstein published an addendum to his seminal STR paper, entitled with "Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?". Based on the conservation of energy he gives the first of his derivations of the formula $E = \Delta m \cdot c^2$. Our presentation is a little bit simpler, we let the direction of relative speed coincide with the direction the energy quanta are emitted. So we can use the formula (1.4) for the longitudinal Doppler shift.

Let be the same situation as in section 12 : A body at rest in system S with mass m_0 emits simultaneously two quanta of radiation, both with energy $L/2$. After emission the body's mass is m_1 . Since the momenta of the emitted quanta add up to zero (or due to a symmetry argument ...) the body stays at rest in frame S.

In the rest frame S of the body conservation of energy means $E_0 = E_1 + L$

Now let be a system S' moving along the path of the emitted quanta with relative speed v as measured in S. The emitting body has speed v in system S' before **and** after the emission. Respecting conservation of energy in S' and using (9.3) and the Doppler shift formula (1.4) we get

$$E_0{}' = E_1{}' + \frac{L}{2} \cdot \left(\sqrt{\frac{(c+v)}{(c-v)}} + \sqrt{\frac{(c-v)}{(c+v)}} \right) = E_1{}' + \frac{L}{2} \cdot \left(\frac{(c+v) + (c-v)}{\sqrt{c^2 - v^2}} \right) = E_1{}' + L \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Hence

$$(E_0{}' - E_1{}') - (E_0 - E_1) = (E_0{}' - E_0) - (E_1{}' - E_1) = L \cdot \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

The interpretation of $(E_0{}' - E_0)$ is kinetic energy $K_0{}'$ of the body in system S' *before* the emission of energy, and $(E_1{}' - E_1)$ is kinetic energy $K_1{}'$ of the body in system S' *after* the emission of energy. The difference of these kinetic energies is given by

$$K_0{}' - K_1{}' = L \cdot \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) = L \cdot \left(1 + \frac{1}{2} \cdot \frac{v^2}{c^2} + \frac{3}{8} \cdot \frac{v^4}{c^4} + \frac{15}{48} \cdot \frac{v^6}{c^6} + \ldots + (-1) \right)$$

There is a loss of kinetic energy *although the velocity of the body remained the same* ! This implies that the emission of energy goes along with a decrease in mass. Neglecting the terms of higher order we find for small velocities v (and v is allowed to be as small as you like !)

$$\frac{1}{2} \cdot (m_0 - m_1) \cdot v^2 = L \cdot \frac{1}{2} \cdot \frac{v^2}{c^2} \qquad \text{or} \qquad \boldsymbol{\Delta m = L / c^2}$$

Einstein writes: "Gibt ein Körper die Energie L in Form von Strahlung ab, so verkleinert sich seine Masse um L / c^2. Hierbei ist offenbar unwesentlich, dass die dem Körper entzogene Energie gerade in Energie der Strahlung übergeht, so dass wir zu der allgemeineren Folgerung geführt werden: Die Masse eines Körpers ist ein Mass für dessen Energie-inhalt."

14. Length Contraction

Let the coordinate systems of "Red" and "Black" move against each other with velocity v respectively $-v$. Let as usual their x-axes fall together and their y- and z-axes be parallel, and let v be parallel to the x-axis.

Black marks two points A and B along his x-axis and measures their distance Δx with his yard stick or with a clock at A and a mirror positioned at B. Then, Black measures (with two synchronized clocks) the time Δt it takes Red to cover the distance from A to B. Now Black calculates their relative speed $v = \Delta x \,/\, \Delta t$.
How will Red measure the distance between A and B on Black's x-axis ? First, Red has to measure the relative speed v of the reference frames in the same way as Black did: Red measures (with two synchronized clocks) the time it takes for the point A to cover a well known distance on his x'-axis. For symmetry reasons, Black and Red agree on the absolute value of v ! Second, Red measures the duration $\Delta t'$ of his flight from A to B. Finally Red calculates $\Delta x' = v \cdot \Delta t'$. So we have

$$\frac{\Delta x}{\Delta t} = v = \frac{\Delta x'}{\Delta t'}$$

For distances in the direction of relative speed we find, using the time dilation formula (1.3)

$$\frac{\Delta x'}{\Delta x} = \frac{\Delta t'}{\Delta t} = \sqrt{1 - \frac{v^2}{c^2}}$$

Red calculates a *shorter* length $\Delta x'$ for the fast moving line segment AB than Black does:

$$\boldsymbol{\Delta x' = \Delta x \cdot \sqrt{1 - \frac{v^2}{c^2}}} \tag{14.1}$$

Black measures the *rest length* or the *eigen length* of the line segment AB. The *rest length* always is the longest to be measured.

Length contraction or *Lorentz contraction* is an immediate consequence of time dilation !
Fast watches are running slow, and fast yard sticks are becoming short.
If Newton's Absolute Time must die then his Absolute Space must share the same fate.

Measuring lenghts in directions perpendicular to relative speed is not affected by length contraction ! Epstein's argument goes as follows: If there were something like *transverse contraction* the track width should shrink with increasing speed of the train - as seen from the train system. As seen from the track system the wheel gauge of the train should shrink with increasing relative speed. So the track would be too wide and too narrow at the same time, and that is impossible. Hence there is no such thing as 'transverse contraction':

$$\boldsymbol{\Delta y' = \Delta y} \qquad \text{and} \qquad \boldsymbol{\Delta z' = \Delta z} \tag{14.2}$$

Compare the corresponding chapters in 'EEE':
https://www.relativity.li/en/epstein2/read/b0_en/b3_en
https://www.relativity.li/en/epstein2/read/b0_en/b4_en
https://www.relativity.li/en/epstein2/read/b0_en/b5_en

15. Desynchronisation

Clocks in different frames of reference do not tick at the same rate, hence it does not make sense to try to synchronize them. However, it is possible to synchronize clocks resting in the same frame. Doing this means to *define* time in that frame ! More details about that in https://www.relativity.li/en/epstein2/read/b0_en/b1_en .

Now let Black synchronize his clocks in his system S (t,x,y,z), and let Red do the same in his system S' (t',x',y',z'). Then both know their own clocks to run synchronously - and both observe that the clocks in the other system are desynchronized in a very specific way !!
For Black, two clocks in Red's frame separated along the x'-axis by the eigen distance $\Delta x'$ are desynchronized by

$$\Delta \boldsymbol{t}' \;=\; -\Delta \boldsymbol{x}' \cdot \frac{\boldsymbol{v}}{\boldsymbol{c}^2} \tag{15.1}$$

or

$$\Delta \boldsymbol{t}' \cdot \boldsymbol{c} \;=\; -\Delta \boldsymbol{x}' \cdot \frac{\boldsymbol{v}}{\boldsymbol{c}} \tag{15.2}$$

The factor c leftside in the second formula merely converts time intervals into distances. Therefore, desynchronisation is proportional to the eigen distance of the clocks along the direction of relative speed and to the quotient of v and c . The minus sign says that clocks that are in an ahead position fall back in time (as seen by Black !). No wonder, these clocks did run away from the sync pulse ... see https://www.relativity.li/en/epstein2/read/b0_en/b6_en .
For Red however all of his own clocks are perfectly synchronized.

The starting point of Einstein's analysis of time was to realize that two events A and B occure simultanously for one observer, while A happens before B for another and B happens before A for a third one ! The statement "these two clocks run synchronously" is not an objective fact given for all observers, but rather a statement that may be true in some frame of reference and false in another. "It's About Time" is the title of Mermin's beautiful book on STR ...

A short derivation of the above formula is presented on https://www.relativity.li/en/epstein2/read/b0_en/b6_en .

Desynchronisation is the third basic phenomenon STR introduces for measurements of time intervals and space intervals (the others are *time dilation* and *length contraction*). Measurements in STR without regarding desynchronisation would quickly lead to contradictions. How is it possible that everybody sees the clocks of the others running slow? How does one avoid the chain of inequalities $\Delta t' \;<\; \Delta t \;<\; \Delta t'$?? Many of the innumerous 'falsifications' of STR are based on this logical short circuit.

It is necessary to take into account the desynchronisation of a set of fast clocks if you want to combine all measurements in two different frames of reference without contradictions. Unfortunately most of the text books avoid the topic. The next section presents a sample problem to show precisely how time dilation, length contraction and desynchronisation work together for complete and consistent relative measurement.

16. Sample Problem for STR Kinematics

The following sample problem shows clearly how time dilation, length contraction and desynchronisation *together* give a complete picture about measurements in different frames of reference.

In Black's laboratory there is a pipe of 12 m length at rest. The pipe is equipped at both ends with detectors/clocks. Now let a particle fly with velocity $v = 0.8 \cdot c$ through that pipe. The rest frame of the particle is called the Red system. Let us answer the following questions:

1. What does Black say about the time it takes the particle to travel through the pipe ?
2. What does Black say about the corresponding time interval in Red's system ?
3. How long is the pipe as seen by Red ?
4. What does Red say about the time interval it takes the pipe to fly over the particle ?
5. What does Red say about the time passing by on each of Black's clocks during that flight ?
6. What is Red's explanation of Black's time measurement ?

Most text books carefully avoid asking questions 5 and 6. Without introducing desynchronisation question 5 causes much confusion and question 6 cannot be answered at all.

However, all of the above questions are easy to answer. For short we write √ for the well known root term

$$\surd \; = \; \sqrt{1 - \frac{v^2}{c^2}} \; = \; \sqrt{1 - 0.8^2} \; = \; 0.6$$

1. Time is distance divided by velocity: $\Delta t \; = \; \Delta x / v \; = \; 12 \text{ m} \;/\; (0.8 \cdot 3 \cdot 10^8 \text{ m/s}) \; = \; 50 \text{ ns}$
2. Due to time dilation Red will measure a shorter duration: $\Delta t' = \Delta t \cdot \surd \; = 50 \text{ ns} \cdot 0.6 = 30 \text{ ns}$
3. Red sees the pipe as length contracted: $\Delta x' = \Delta x \cdot \surd \; = 12 \text{ m} \cdot 0.6 = 7.2 \text{ m}$
4. Time is distance divided by velocity: For the flight of the pipe over Red with speed v it takes
   $\Delta t' \; = \; \Delta x' \;/\; v \; = \; 7.2 \text{ m} \;/\; (0.8 \cdot 3 \cdot 10^8 \text{ m/s}) \; = \; 30 \text{ ns}$
   Black and Red are in complete agreement about Red's measurement !
5. Black's fast clocks tick slow as seen by Red. During the 30 ns passing on Red's clock, on each of Black's clocks passes the time $\Delta t \; = \; \Delta t' \cdot \surd \; = \; 30 \text{ ns} \cdot 0.6 \; = \; 18 \text{ ns}$!!
6. However, Red can calculate the time Black measures in this experiment. Black's clocks are (as seen by Red) desynchronised by $\Delta t \; = \; \Delta x \cdot v \;/\; c^2 \; = \; 12 \text{ m} \cdot 0.8 \;/\; (3 \cdot 10^8 \text{ m/s}) \; = \; 32 \text{ ns}$. Black's rear clock is 32 ns ahead ! Together with the 18 ns of the 'actual' duration Black will measure 50 ns using his two clocks positioned at both ends of the pipe.

Black needs two distant clocks for his measurements, one at each end of the pipe. Their synchronization is no objective fact given to all observers ! Both Red and Black are able to calculate the results of the measurements of the other. Their calculations are in complete agreement with the effective measurements. The measured values differ, but they are not contradictory. The values are 'relative', but not arbitrary.

17. Transverse Velocities and Transversal Doppler Shift

Let the inertial frames of "Red" and "Black" move against each other with relative speed v, with their axes oriented as usual and v being parallel to the x- and the x'-axes.

In Red's frame S' (t',x',y',z') an object is moving with speed u' along the y'-direction. How calculates Black the y-component u of the velocity that object has in his frame S (t,x,y,z) ?

We have $u = \Delta y / \Delta t$, $u' = \Delta y' / \Delta t'$ and $\Delta y = \Delta y'$. Moreover, for Black holds $\Delta t' = \Delta t \cdot \sqrt{1 - \frac{v^2}{c^2}}$

Hence we get

$$\boldsymbol{u} = \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\Delta t} = \frac{\Delta y'}{\Delta t'} \cdot \sqrt{1 - \frac{v^2}{c^2}} = \boldsymbol{u}' \cdot \sqrt{\boldsymbol{1} - \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}} \qquad (17.1)$$

For Black, the *transverse velocity* of the object is slowed down by the well known root factor. All processes are slowed down in Red's world as seen by Black ! Let us look back to the transversal Doppler shift in this context:

Red crosses in some distance Δx the x-axis of Black with velocity v in y-direction. For the moment the distance of Red and Black does not change. But any oscillator in Red's frame undergoes time dilation as seen by Black. If Red is sending in his frame with frequency f' Black receives waves with frequency

$$\boldsymbol{f} = \boldsymbol{f}' \cdot \sqrt{\boldsymbol{1} - \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}} \qquad (1.6) \equiv (17.2)$$

Once again, this is the formula for the transversal Doppler shift. For Black all processes in Red's frame appear delayed ...

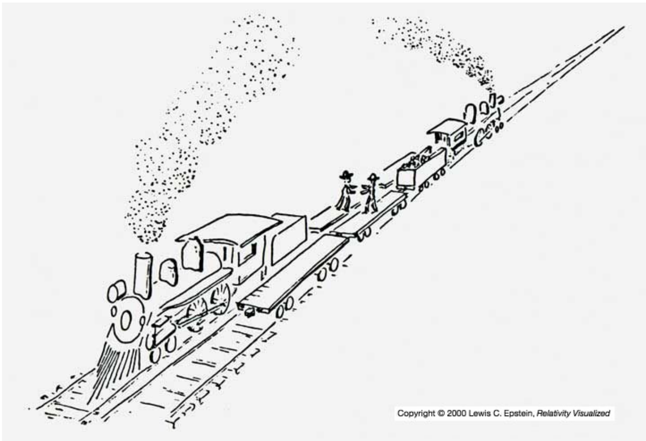
... and for Red things are just the other way round !

We will use formula (17.1) for transversal speed in the next section to derive the STR expression of momentum with a absolute minimum of assumptions.

18. Derivation of Relativistic Momentum Based on a Minimum of Assumptions

In section 4 we derived the STR formula of momentum striving for conservation of momentum and conservation of mass. The derivation presented in this section neither assumes conservation of mass nor conservation of momentum. The equation for the momenta is given by the symmetry of the arrangement.
The same presentation can be found at https://www.relativity.li/en/epstein2/read/e0_en/e1_en or in the book "Die Relativitätstheorie Einsteins" of Max Born (first edition 1920, enhanced editions 1964 and 1969).

The identical twins Peter and Danny (Epstein's nephews ...) exchange completely symmetric punches standing on platforms of two Einstein trains :



The relative speed of the trains is v , both fists have the same rest mass m_0 and both young men are punching with the same speed u transverse to the velocity of the train (as measured by themselves!). Due to symmetry both momenta of their fists in direction of their punch add up to zero :

$$p_y(Peter) \;=\; -\,p_y(Danny)$$

For Peter the transverse velocity u' of Danny's fist is slowed down following (17.1). He wonders why Danny could hit him so hard by his slow hand. If Danny is not hiding some additional mass in his fist we have to suspect mass might depend on relative speed. So, Peter writes down the following equation for the momenta in the y-direction:

$$m_u \cdot u \;=\; -\,m_{v+u'} \cdot u' \;=\; -\,m_{v+u'} \cdot (-u) \cdot \sqrt{1 - \frac{v^2}{c^2}} \qquad (18.1)$$

Let (interim) $v + u'$ stand for the velocity of Danny's fist as seen in Peter's frame. Dividing by u we get

$$m_u \;=\; m_{v+u'} \cdot \sqrt{1 - \frac{v^2}{c^2}}$$

This holds for arbitrarily small velocities u ! So it is true in the limit of $u' \rightarrow 0$. Then we have $u = 0$, m_u turns into m_0 , $m_{v+u'}$ into m_v and we get the equations (4.1) and (4.2) :

$$\boldsymbol{m_v \;=\; \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \;=\; \gamma \cdot m_0} \qquad \text{and} \qquad \boldsymbol{p \;=\; m_v \cdot v \;=\; \gamma \cdot m_0 \cdot v} \qquad (18.2)$$

19. Conservation of Momentum Implies Conservation of Dynamic Mass

In section 18 we derived the formulas (4.1) and (4.2) of dynamic mass and STR momentum based from nothing else than time dilation. We did not use conservation of momentum nor conservation of any type of mass. In this section we deduce conservation of dynamic mass from conservation of momentum. The line of arguments and the figures are taken from "The Wonderful World of Relativity" by Andrew M. Steane (Oxford University Press 2011) :

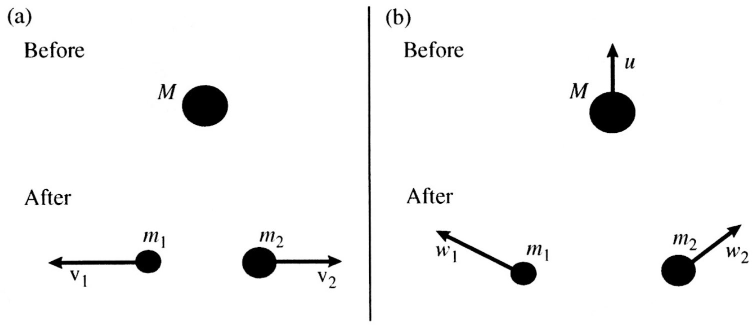


Figure (a) of the figure shows a mass M at rest, disintegrating into two pieces with rest mass m_1 and m_2 . The pieces will fly into opposite directions due to conservation of momentum. Let the velocities v_1 and v_2 be horizontal. Figure (b) shows the same disintegration as observed in a frame moving with speed u downwards. With (18.2) conservation of the vertical component of momentum means

$$M \cdot u \cdot \gamma_u \;=\; m_1 \cdot u \cdot \gamma_{w_1} \;+\; m_2 \cdot u \cdot \gamma_{w_2}$$

Dividing by u we get

$$M \cdot \gamma_u \;=\; m_1 \cdot \gamma_{w_1} \;+\; m_2 \cdot \gamma_{w_2}$$

The equation is true for arbitrarily small values of u . In the limit $u \to 0$ we get

$$M \;=\; m_1 \cdot \gamma_{v_1} \;+\; m_2 \cdot \gamma_{v_2} \qquad (19.1)$$

(19.1) expresses conservation of dynamic mass. Rest mass is **not** conserved, the values of γ_{v_1} and γ_{v_2} are greater than 1 ; we have $M > m_1 + m_2$.

Still following Andrew M. Steane we rewrite equation (19.1) :

$$\boldsymbol{M \;=\; m_1 \;+\; m_1 \cdot (\gamma_{v_1} - 1) \;+\; m_2 \;+\; m_2 \cdot (\gamma_{v_2} - 1)} \qquad (19.2)$$

On the right side, after the splitting, total dynamic mass is composed of the rest masses and two small additional masses. Multiplying (19.2) by c^2 turns conservation of dynamic mass into conservation of total energy:

$$\boldsymbol{M \cdot c^2 \;=\; m_1 \cdot c^2 \;+\; m_1 \cdot c^2 \cdot (\gamma_{v_1} - 1) \;+\; m_2 \cdot c^2 \;+\; m_2 \cdot c^2 \cdot (\gamma_{v_2} - 1)} \qquad (19.3)$$

On the right side, total energy is given by the sum of the rest energies and the kinetic energies. If you accept the idea of rest energy this argument provides another derivation of STR kinetic energy. No forces, no work performed, no integral, just conservation of momentum!

In reversed time order the above process is known as perfectly inelastic collision and is thoroughly discussed in the book "Die Relativitätstheorie Einsteins" published by Max Born in 1920 (still available by Springer 1964 and later).

20. Deriving the Lorentz-Transformations

Let the inertial frames of "Red" and "Black" move against each other with relative speed v . Let the coordinate frames be oriented as usual, with the x-axes along the same straight line and the y-axes and z-axes being parallel. Black labels events in his frame S with coordinates (*t,x,y,z*) , Red does the same in his system S' using coordinates (*t',x',y',z'*) . How can we calculate the coordinates of a specific event in frame S if we know its coordinates in S' ?

Following (14.2) there is no 'transverse contraction', and so we have

$$\boldsymbol{y} = \boldsymbol{y}' \qquad \text{und} \qquad \boldsymbol{z} = \boldsymbol{z}' \tag{20.1}$$

To find the transformations of the t'- and x'- values we need an additional agreement: Both Red and Black reset their master clocks positioned at (0,0,0) at the very moment their coordinate frames coincide. Afterwards, Red and Black synchronised all the other clocks in their own frame with that master clock. Both preconditions are necessary, because it does not make sense to compare measured coordinates of a single event. We can only compare *intervals* of time and *intervals* of space. For transforming time- and space-coordinates we need an *event of reference* or a common *origin-event*. Later events are labelled by their distance (in time and space) to that event of reference.

All of that given, let Red ascribe the labels (t', x') to a specific event. For Black, the Red clock positioned at x' is desynchronised against Red's masterclock in Red's origin. According to (15.1) the Red master clock showed

$$t' + \frac{x' \cdot v}{c^2}$$

when that event occurred. But, like all clocks in Red's frame, that master clock runs slow as seen by Black. Therefore Black calculates the time *his* clocks showed when that event happened by

$$\boldsymbol{t} = \frac{\boldsymbol{t}' + \frac{\boldsymbol{x}' \cdot \boldsymbol{v}}{\boldsymbol{c}^2}}{\sqrt{\mathbf{1} - \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}}} \tag{20.2}$$

What is Black's calculation of the position x of that event in his frame ? The position of Red's origin is given by $O_{red} = v \cdot t$. For Black, Red's measurement of the distance x' of that event from is influenced by Lorentz contraction. So Black calculates

$$\boldsymbol{x} = v \cdot t + \frac{x'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{v \cdot t'}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{x'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\boldsymbol{x}' + \boldsymbol{v} \cdot \boldsymbol{t}'}{\sqrt{\mathbf{1} - \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}}} \tag{20.3}$$

Time and space coordinates cannot be separated any longer.

For the reverse transformations we just have to replace v by $-v$, the roles of Red and Black are completely symmetric.

Henri Poincaré has named this group of coordinate transformations 'Lorentz-Transformations'. Hendrik Antoon Lorentz introduced them shortly before 1900 to handle the contradictions arising from a constant speed of light and a resting ether system. Poincaré further showed that these transformations constitute a group in the sense of mathematical group theory.

Let us write down the Lorentz-transformations for both directions:

$$\boldsymbol{t} = \frac{\boldsymbol{t'} + \dfrac{\boldsymbol{x'} \cdot \boldsymbol{v}}{\boldsymbol{c^2}}}{\sqrt{\boldsymbol{1} - \dfrac{\boldsymbol{v^2}}{\boldsymbol{c^2}}}} \qquad \boldsymbol{t'} = \frac{\boldsymbol{t} - \dfrac{\boldsymbol{x} \cdot \boldsymbol{v}}{\boldsymbol{c^2}}}{\sqrt{\boldsymbol{1} - \dfrac{\boldsymbol{v^2}}{\boldsymbol{c^2}}}}$$

$$\boldsymbol{x} = \frac{\boldsymbol{x'} + \boldsymbol{v} \cdot \boldsymbol{t'}}{\sqrt{\boldsymbol{1} - \dfrac{\boldsymbol{v^2}}{\boldsymbol{c^2}}}} \qquad \boldsymbol{x'} = \frac{\boldsymbol{x} - \boldsymbol{v} \cdot \boldsymbol{t}}{\sqrt{\boldsymbol{1} - \dfrac{\boldsymbol{v^2}}{\boldsymbol{c^2}}}}$$

$$\boldsymbol{y} = \boldsymbol{y'} \qquad \boldsymbol{y'} = \boldsymbol{y}$$

$$\boldsymbol{z} = \boldsymbol{z'} \qquad \boldsymbol{z'} = \boldsymbol{z} \qquad (20.4)$$

We will need these transformations in the next section to calculate differential terms like $dx' \,/\, dt$ or $dt \,/\, dt'$.

Newton's Absolute Time requests $t = t'$, together with his Absolute Space we further have $x = x' + v \cdot t'$ and $x' = x - v \cdot t$. These are the well known Galilei transformations. They follow from the Lorentz transformations in the limit of $c \rightarrow \infty$. The mere *existence of a limiting speed* is not compatible with Newton's Absolute Time and Absolute Space, it forces the developement of STR !

Let us note these transformations by using the abbreviations β_v and γ_v as defined in (6.2) and (6.3) :

$$\boldsymbol{t} = \boldsymbol{\gamma_v} \cdot \left(\boldsymbol{t'} + \boldsymbol{\beta_v} \cdot \frac{\boldsymbol{x'}}{\boldsymbol{c}}\right) \qquad \boldsymbol{t'} = \boldsymbol{\gamma_v} \cdot \left(\boldsymbol{t} - \boldsymbol{\beta_v} \cdot \frac{\boldsymbol{x}}{\boldsymbol{c}}\right)$$

$$\boldsymbol{x} = \boldsymbol{\gamma_v} \cdot (\boldsymbol{x'} + \boldsymbol{\beta_v} \cdot \boldsymbol{c} \cdot \boldsymbol{t'}) \qquad \boldsymbol{x'} = \boldsymbol{\gamma_v} \cdot (\boldsymbol{x} - \boldsymbol{\beta_v} \cdot \boldsymbol{c} \cdot \boldsymbol{t})$$

$$\boldsymbol{y} = \boldsymbol{y'} \qquad \boldsymbol{y'} = \boldsymbol{y}$$

$$\boldsymbol{z} = \boldsymbol{z'} \qquad \boldsymbol{z'} = \boldsymbol{z} \qquad (20.5)$$

Multiplying the equations for t and t' by c we get equations for the new variables $c \cdot t$ and $c \cdot t'$. These are formally identical to those for the variables x and x' ! The same effect has the choice of units for time and space measurement so that the speed of light becomes 1 .

The full symmetry group of STR results when you add the rotations of space to the Lorentz transformations. The resulting group is called the Poincaré group. Poincaré also proved that group to be the symmetry group of Maxwell's theory.

Lorentz transformations handle only coordinate frames in special orientation to each other (the x-axes coinciding, the y- and z-axes being parallel and relative speed v running along the x-axes). This special situation is frequently called a 'Lorentz boost'. But is this situation relly so special ? We are free to choose our coordinate frames and why should we not choose one which makes our calculations as simple as possible ?

21. The Addition of Arbitrary Velocities

Let us derive the formulas for the transformation of arbitrary velocities from the Lorentz transformations.

Let the inertial frames of Red and Black be oriented as usual. Red moves with relative speed $\boldsymbol{v}$ along the x-axis of Black. In Red's system some object moves with velocity $\boldsymbol{u}'$ in arbitrary direction. What is the velocity $\boldsymbol{u}$ of that object in Black's frame ?

We use the notations

$$\boldsymbol{v} = (v, 0, 0) \quad , \quad \boldsymbol{u}' = (u_x', u_y', u_z') = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}\right) \quad \text{and} \quad \boldsymbol{u} = (u_x, u_y, u_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

We calculate the components of $\boldsymbol{u}$ by using differential terms calculated from the formulas (20.5) of the last section :

$$u_x = \frac{dx}{dt} = \frac{dx}{dt'} \cdot \frac{dt'}{dt} = \frac{\frac{dx}{dt'}}{\frac{dt}{dt'}} = \frac{\gamma \cdot \left(\frac{dx'}{dt'} + \beta \cdot c \cdot \frac{dt'}{dt'}\right)}{\gamma \cdot \left(\frac{dt'}{dt'} + \beta \cdot \frac{1}{c} \cdot \frac{dx'}{dt'}\right)} = \frac{u_x' + \frac{v}{c} \cdot c \cdot 1}{1 + \frac{v}{c} \cdot \frac{1}{c} \cdot u_x'} = \frac{u_x' + v}{1 + \frac{v \cdot u_x'}{c^2}} \tag{21.1}$$

This is formula (2.1) again!

In a similar way we calculate the terms for u_y and u_z :

$$u_y = \frac{dy}{dt} = \frac{dy}{dt'} \cdot \frac{dt'}{dt} = \frac{\frac{dy}{dt'}}{\frac{dt}{dt'}} = \frac{\frac{dy'}{dt'}}{\gamma \cdot \left(\frac{dt'}{dt'} + \beta \cdot \frac{1}{c} \cdot \frac{dx'}{dt'}\right)} = \frac{u_y'}{\gamma \cdot \left(1 + \frac{v}{c} \cdot \frac{1}{c} \cdot u_x'\right)} = \frac{u_y'}{\gamma \cdot \left(1 + \frac{v \cdot u_x'}{c^2}\right)} \tag{21.2}$$

$$\text{and in the same manner} \qquad u_z = \dots \dots = \frac{u_z'}{\gamma \cdot \left(1 + \frac{v \cdot u_x'}{c^2}\right)} \tag{21.3}$$

The same formulas enable us to calculate $\boldsymbol{u}'$ from $\boldsymbol{v}$ and $\boldsymbol{u}$ if $\boldsymbol{v}$ is replaced by $-\boldsymbol{v}$ **.**

Now let us suppose the z-component of $\boldsymbol{u}'$ to be zero. Hence the z-component of $\boldsymbol{u}$ is zero, too. This is no limitation of generality: By rotation of the frames S and S' around the x-axis you can always arrange the x-y-plane to fall together with the plane defined by $\boldsymbol{v}$ and $\boldsymbol{u}'$.

Let us denote by α' the angle between $\boldsymbol{u}'$ and $\boldsymbol{v}$. With $u_z' = 0$ by arrangement we have

$$tan(\alpha') = \frac{u_y'}{u_x'} \tag{21.4}$$

We calculate the angle α between $\boldsymbol{u}$ and $\boldsymbol{v}$ using (21.1) and (21.2) :

$$tan(\alpha) \;=\; \frac{u_y}{u_x} \;=\; \frac{\dfrac{u_y{}'}{\gamma \cdot \left(1 + \dfrac{v \cdot u_x{}'}{c^2}\right)}}{\dfrac{u_x{}' \;+\; v}{\left(1 + \dfrac{v \cdot u_x{}'}{c^2}\right)}} \;=\; \frac{u_y{}' \cdot \sqrt{1 - \dfrac{v^2}{c^2}}}{u_x{}' \;+\; v} \qquad (21.5)$$

A positive velocity v implies the numerator decreased and the denominator increased. In that case we find $tan(\alpha) \;<\; tan(\alpha')$.

Einstein started with

$$\boldsymbol{u}^2 \;=\; (u_x)^2 \;+ \left(u_y\right)^2 \quad , \quad \boldsymbol{u}'^2 \;=\; (u_x{}')^2 \;+ \left(u_y{}'\right)^2 \qquad \text{and} \qquad tan(\alpha') \;=\; \frac{u_y{}'}{u_x{}'}$$

and found "by a simple calculation"

$$\boldsymbol{u} \;=\; \frac{\sqrt{(v^2 \;+ u'^2 \;+ 2 \cdot v \cdot u' \cdot cos\;\alpha'\,) \;-\; \left(\dfrac{v \cdot u' \cdot sin\;\alpha'}{c}\right)^2}}{1 + \dfrac{v \cdot u_x{}' \cdot cos\;\alpha'}{c^2}} \qquad (21.6)$$

Einstein writes: "Es ist bemerkenswert, dass $\boldsymbol{v}$ und $\boldsymbol{u}'$ in symmetrischer Weise in den Ausdruck für die resultierende Geschwindigkeit eingehen. Hat auch $\boldsymbol{u}'$ die Richtung der x-Achse so erhalten wir ..." ... formula (2.1) again. Then we have $cos(\alpha') = 1$ and $sin(\alpha') = 0$.

Indeed, using (21.1) and (21.2) Einstein's "simple calculation" can be done without difficulties.

Let us derive formula (17.1) again in a more sophisticated way. For the transverse velocity $\boldsymbol{u}' \;=\; (\,0\,, u_y{}'\,, 0\,)$ we have according to (21.2)

$$\boldsymbol{u_y} \;=\; \frac{u_y{}'}{\gamma \cdot \left(1 + \dfrac{v \cdot u_x{}'}{c^2}\right)} \;=\; \frac{u_y{}'}{\gamma \cdot \left(1 + \dfrac{v \cdot 0}{c^2}\right)} \;=\; \boldsymbol{u_y}' \cdot \sqrt{\boldsymbol{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}} \qquad (17.1) \quad \equiv \quad (21.7)$$

22. Aberration

The formulas of the last section hold for any velocities $\boldsymbol{u}$ and $\boldsymbol{u}'$, and they are still valid for $\boldsymbol{u} = \boldsymbol{c}$, the speed of light. So let the light of a distant star arrive intersecting at angle α with our x-axis. The components of that speed are

$$u_x = -c \cdot cos\, \alpha \quad , \quad u_y = c \cdot sin\, \alpha \quad \text{and} \quad u_z = 0$$

By our choice, α is an acute angle for stars with positive x-values. Following formulas (21.1) and (21.2) the velocity of the light of that star has for Red the components

$$u_x{}' = \frac{-c \cdot cos\, \alpha - v}{1 + \frac{-v \cdot (-c) \cdot cos\, \alpha}{c^2}} = \frac{-c \cdot cos\, \alpha - v}{1 + \frac{v \cdot cos\, \alpha}{c}}$$

$$u_y{}' = \frac{c \cdot sin\, \alpha}{\gamma \cdot \left(1 + \frac{-v \cdot (-c) \cdot cos\, \alpha}{c^2}\right)} = \frac{c \cdot sin\, \alpha}{\gamma \cdot \left(1 + \frac{v \cdot cos\, \alpha}{c}\right)}$$

Of course, $(u_x{}')^2 + (u_y{}')^2 = c^2$ holds for these components, too.

For Red in S', the light from this star builds an acute angle α' to the x'-axis with

$$\boldsymbol{tan\, \alpha'} = \frac{u_y{}'}{-u_x{}'} = \frac{\frac{c \cdot sin\, \alpha}{\gamma \cdot \left(1 + \frac{v \cdot cos\, \alpha}{c}\right)}}{\frac{c \cdot cos\, \alpha + v}{\left(1 + \frac{v \cdot cos\, \alpha}{c}\right)}} = \frac{sin\, \alpha}{\gamma \cdot \left(cos\, \alpha + \frac{v}{c}\right)} = \frac{\boldsymbol{sin\, \alpha}}{\boldsymbol{\gamma \cdot (cos\, \alpha + \beta)}} \tag{22.1}$$

Einstein preferred in his publication another formula :

$$\boldsymbol{cos\, \alpha'} = \frac{-u_x{}'}{c} = \frac{cos\, \alpha + \frac{v}{c}}{1 + \frac{v \cdot cos\, \alpha}{c}} = \frac{\boldsymbol{cos\, \alpha + \beta}}{\boldsymbol{1 + \beta \cdot cos\, \alpha}} \tag{22.2}$$

Einstein comments: "Diese Gleichung drückt das Aberrationsgesetz in seiner allgemeinsten Form aus." Because he uses angle $\varphi = 180° - \alpha$ instead of α his cosine values have the opposite signs. In the special case of $\alpha = 90°$ we have $cos \propto ' = \beta = v/c$. For the difference δ' to 90° degree we have $sin\, \delta' = cos\, \alpha' = v/c$.
Until 1905 astronomers used the 'wrong' formula $tan\, \delta' = v/c$ resulting from the 'old' way to add velocities. However for small angles the difference between those formulas is far less than what can be observed.

Using the goniometric identity

$$tan\, \frac{\alpha}{2} = \frac{sin\, \alpha}{1 + cos\, \alpha}$$

the relation between α and α' can be expressed by a beautyful symmetric formula. With (21.1) and (21.2) we find

$$\tan \frac{\alpha'}{2} = \frac{\sin \alpha'}{1 + \cos \alpha'} = \frac{\frac{u_y{}'}{c}}{1 - \frac{u_x{}'}{c}} = \frac{\frac{sin\, \alpha}{\gamma \cdot (1 + \beta \cdot cos\, \alpha)}}{1 + \frac{cos\, \alpha + \beta}{1 + \beta \cdot cos\, \alpha}} = \frac{sin\, \alpha}{\gamma \cdot (1 + \beta \cdot cos\, \alpha + cos\, \alpha + \beta)} =$$

$$= \frac{sin\, \alpha}{\gamma \cdot (1 + \beta) \cdot (1 + cos\, \alpha)} = \frac{1}{\gamma \cdot (1 + \beta)} \cdot \frac{sin\, \alpha}{1 + cos\, \alpha} = \frac{\sqrt{1 - \beta^2}}{(1 + \beta)} \cdot tan\, \frac{\alpha}{2} =$$

$$= \frac{\sqrt{1-\beta}\cdot\sqrt{1+\beta}}{\sqrt{1+\beta}\cdot\sqrt{1+\beta}} \cdot tan\,\frac{\alpha}{2} \;=\; \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \cdot tan\,\frac{\alpha}{2} \;=\; \sqrt{\frac{c-v}{c+v}} \cdot tan\,\frac{\alpha}{2}$$

So we get

$$\boldsymbol{tan\,\frac{\alpha'}{2} \;=\; \sqrt{\frac{1-\beta}{1+\beta}} \cdot tan\,\frac{\alpha}{2} \;=\; \sqrt{\frac{c-v}{c+v}} \cdot tan\,\frac{\alpha}{2}} \qquad (22.3)$$

Astronomer "Red", heading towards the star with velocity v observes a smaller angle α' than his fellow "Black" who sits at rest relative to the star (or has a minor relative velocity to it).

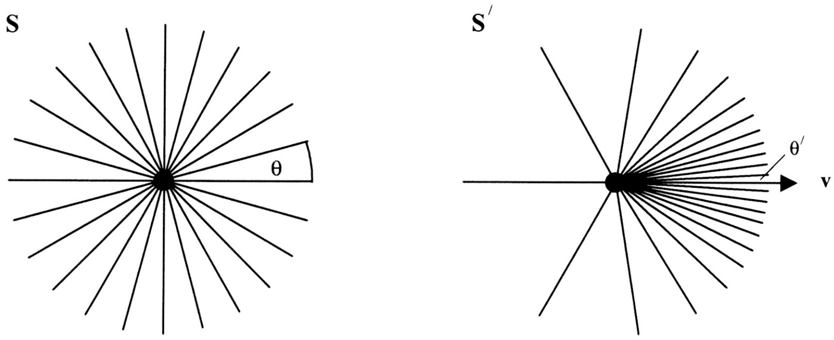
'aberrare' means to err or to deviate. A legend has it that James Bradley became aware of that effect in 1727 while riding in a carriage under the English rain. He observed the rain falling more and more in front the faster the carriage drove. By a finite value of the speed of light, he realized, the same effect should show up.

The speed of earth orbiting the sun is about 30 km/s . If the position of a star is perpendicular to earth's velocity, that is α = 90°, the calculation gives a value of aberration of about 20 arc seconds :

$$tan\,\frac{\alpha'}{2} \;=\; \sqrt{\frac{c-v}{c+v}} \cdot tan\,\frac{90°}{2} \;\approx\; \sqrt{\frac{300'000-30}{300'000+30}} \cdot 1$$

$$\alpha' \;= 2\cdot\frac{\alpha'}{2} \;=\; 2\cdot arc\,tan\left(\sqrt{\frac{300'000-30}{300'000+30}}\right) \;\approx\; 89°\,59'\,39.4"$$

The following figures (adapted for this paper) are taken from the book "Spezielle Relativitätstheorie für Studienanfänger" by Jürgen Freund (vdf Hochschulverlag Zürich 2004). On the left it shows an isotropic radiation falling radially on system S. The right figure shows the same radiation as observed by someone moving to the right with speed $v = 0.9 \cdot c$. In the fast system S' the radiation falls in with greater concentration from ahead, as the rain drops do for Bradley's carriage example:

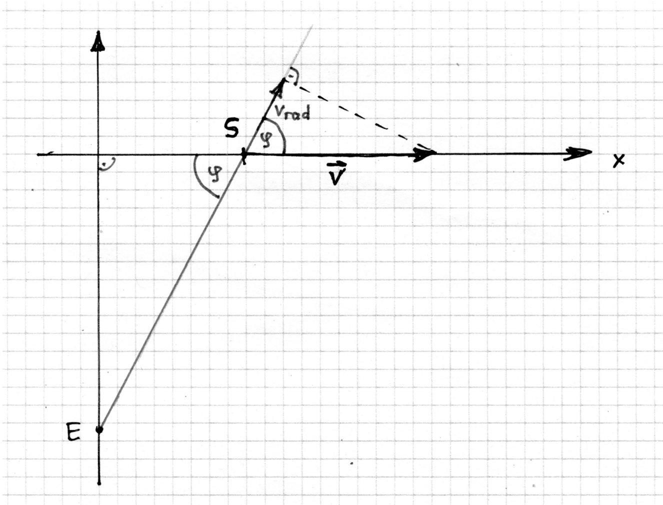


For Red in system S', the radiation is not only concentrated in the forward direction, it is also more intense due to Doppler shift in that direction. Instead of visible light you might be hit by UV-radiation or by x-rays ! And the world 'behind' disappears in darkness ...

Longitudinal and transversal Doppler shift were treated in the first section. For the sake of completeness we will study the general case in the next section.

23. Doppler Shift: The Universal Formula

Let a sender S move with velocity v in an arbitrary direction relative to the receiver E. The sender S is emitting radiation of frequency f_S as measured by himself.



E will receive that radiation at (in this case reduced) frequency f_E for two reasons:

a) Time dilation slows down the oscillator frequency of the sender by the well known root expression (as seen by the receiver). This effect does not depend on the direction of relative movement.

b) The increasing distance to the sender stretches the wave lengths of the radiation according to longitudinal Doppler shift. The amount of this effect depends only on the radial velocity $v_{rad} = v \cdot cos\ \varphi$.

In the general case we can do the same calculation as for (1.4) if we insert v_{rad} for v at the proper position:

$$f_E = f_S \cdot \frac{c}{c + v_{rad}} \cdot r(v) = f_S \cdot \frac{c}{c + v \cdot cos\ \varphi} \cdot \sqrt{1 - \frac{v^2}{c^2}} = f_S \cdot \frac{1}{1 + \frac{v \cdot cos\ \varphi}{c}} \cdot \sqrt{1 - \frac{v^2}{c^2}}$$

or

$$\boldsymbol{f_E = f_S \cdot \frac{\sqrt{c^2 - v^2}}{c + v \cdot cos\ \varphi} = f_S \cdot \frac{1}{\gamma \cdot (1 + \beta \cdot cos\ \varphi)}} \qquad (23.1)$$

Doing the same calculation with angle $\theta = 180° - \varphi$ we get a minus sign in the denominator.

What about the special cases ?

- If φ equals zero sender S moves straight away from receiver E along the line ES . Then we have $v_{rad} = v$ and $cos\ \varphi = 1$ and we repeat the calculation of longitudinal Doppler shift leading to (1.4).

- For $\varphi = 90°$ the sender moves at right angle to the line of sight. Then we have $v_{rad} = 0$ and $cos\ \varphi = 0$ and we get the formula for transverse Doppler shift according to (1.6) or (17.2).

24. Four-Vectors, Three-Vectors and Newtons Second Law

A powerful tool for doing calculations in STR is provided through the use of *four-vectors*. Time coordinate and the three spatial coordinates of events are combined to one vector with four components:

$$X = (c \cdot t, x, y, z) = (c \cdot t, \vec{x})$$

Time is multiplied by c in order to have the same units for all components. X gives us a position in 4-space, X is the *four-vector of position*. Deriving X by time t is not a useful idea because time differs from one inertial frame to the other. The only distinguished time is the time in the rest frame of the moving object, i.e., the so called *proper time* τ . *Four-velocity* V is therefore defined by

$$V = \frac{d}{d\tau}(X) = \frac{d}{dt}(X) \cdot \frac{dt}{d\tau} = \gamma \cdot \frac{d}{dt}(X) = \gamma \cdot (c, \vec{v})$$

Multiplying V by the rest mass m_0 of the moving body we get the *four-momentum*

$$P = m_0 \cdot V = \gamma \cdot m_0 \cdot (c, \vec{v}) = \left(\frac{E_{tot}}{c}, \vec{p}\right) \tag{24.1}$$

The last equation is based on (4.2) and (5.4). So, $\vec{p}$ is the *STR momentum* three-vector !

Further, *four-force* F is defined by

$$F = \frac{d}{d\tau}(P) = \frac{d}{dt}(P) \cdot \frac{dt}{d\tau} = \gamma \cdot \frac{d}{dt}(P) = \gamma \cdot \left(\frac{1}{c} \cdot \frac{dE}{dt}, \frac{d\vec{p}}{dt}\right) = \gamma \cdot \left(\frac{1}{c} \cdot \vec{f} \cdot \vec{v}, \vec{f}\right) \tag{24.2}$$

where $\vec{f}$ denotes the conventional 3d force vector.

We are not going to work with four-vectors here. Our aim is to show which relations of three-vectors remain valid even in STR. The spatial part of (24.2) shows that Newton's second axiom of mechanics survives, seemingly unchanged:

$$\boldsymbol{\vec{f}} = \frac{\boldsymbol{d\vec{p}}}{\boldsymbol{dt}} \tag{24.3}$$

(24.3) needs no proof but gives a *definition* of force $\vec{f}$, as it did before in Newton's theory.The adaptation to STR is hidden in the definition of STR momentum.

The temporal part of equation (24.2) shows that the equation of power (i.e. the rate of change of energy) remains completely unchanged:

$$\frac{dE}{dt} = \vec{f} \cdot \vec{v} = \vec{f} \cdot \frac{d\vec{x}}{dt} \qquad \text{or} \qquad dE = \vec{f} \cdot d\vec{x} \tag{24.4}$$

The right side of (24.4) is the basic definition of energy as performed work or the ability to perform work.

In section 5 we combined (24.3) and (24.4) to calculate the relativistic expression of kinetic energy, integrating

$$\boldsymbol{dE} = \boldsymbol{\vec{f}} \cdot \boldsymbol{\vec{v}} \cdot \boldsymbol{dt} = \frac{d\vec{p}}{dt} \cdot \vec{v} \cdot dt = \frac{d\vec{p}}{dv} \cdot \frac{dv}{dt} \cdot \vec{v} \cdot dt = \frac{\boldsymbol{d\vec{p}}}{\boldsymbol{dv}} \cdot \boldsymbol{\vec{v}} \cdot \boldsymbol{dv} \tag{24.5}$$

The result (5.2) of that calculation became independently confirmed in section 18 .

Let us go one step further: *Four-acceleration* A is defined by

$$A = \frac{d}{d\tau}(V)$$

By definition, the formula $F = m_0 \cdot A$ is correct in STR :

$$F = \frac{d}{d\tau}(P) = \frac{d}{d\tau}(m_0 \cdot V) = m_0 \cdot \frac{d}{d\tau}(V) = m_0 \cdot A$$

The representation of the four-vector A by three-vectors is rather complicated in general. However, the calculation is straightforward, resulting in

$$A = \frac{d}{d\tau}(V) = \gamma^4 \cdot c^{-2} \cdot \vec{v} \cdot \vec{a} \cdot (c, \vec{v}) + \gamma^2 \cdot (0, \vec{a}) \tag{24.6}$$

With $F = m_0 \cdot A$, (24.2) and (24.6) we get

$$\gamma \cdot (\frac{1}{c} \cdot \vec{f} \cdot \vec{v}, \vec{f}) = m_0 \cdot \gamma^4 \cdot c^{-2} \cdot \vec{v} \cdot \vec{a} \cdot (c, \vec{v}) + m_0 \cdot \gamma^2 \cdot (0, \vec{a}) \tag{24.7}$$

Taking a closer look at (24.7) we notice that three-force $\vec{f}$ is not necessarily parallel to three-acceleration $\vec{a}$ in STR !

The first summand on the right side of (24.6) and (24.7) disappears if acceleration and velocity are perpendicular to each other (this is always the case if Lorentz-force is at work). Then (24.7) reduces to

$$\gamma \cdot (\frac{1}{c} \cdot \vec{f} \cdot \vec{v}, \vec{f}) = m_0 \cdot \gamma^2 \cdot (0, \vec{a})$$

and hence $\vec{f} = \gamma \cdot m_0 \cdot \vec{a}$. So, in the early days of STR, people were speaking of 'transversal mass' $\gamma \cdot m_0$.

If $\vec{v}$ and $\vec{a}$ are parallel to each other we get from (24.2) and (5.1)

$$\vec{f} = \frac{d\vec{p}}{dt} = \frac{d\vec{p}}{dv} \cdot \frac{dv}{dt} = \gamma^3 \cdot m_0 \cdot \vec{a}$$

The same formula results, of course, by a direct calculation starting at (24.7). And here we have the origin of the outdated notion 'longitudinal mass' for the term $\gamma^3 \cdot m_0$.

In modern text books on STR the term 'mass' exclusively denotes rest mass m_0 . I take the liberty to carry on using the term 'dynamic mass' for $\gamma \cdot m_0$. After all it is a *conserved quantity*, proportional to total energy and hence to the inertia of the object. On the other hand, rest mass m_0 is a relativistic *invariant*, it has the same value in all frames of reference. But it is not a conserved quantity.

The terms 'longitudinal mass' and 'transversal mass' however are definitively obsolete.

25. Some Remarks on the Foundations of STR

STR is the only possible way to unite the principle of relativity and Maxwell's theory of electromagnetism. In Maxwell's theory electromagnetic waves are spreading through empty space with universal speed c in all frames of reference, independent of any movement of the observer or the source - something completely incompatible with Newton's concepts of Absolute Time and Absolute Space. Lorentz tried to reconcile Maxwell's and Newton's theories introducing a 'length contraction' and a 'local time'. Till 1900 he developed a great part of the mathematical tools for the yet to come STR.

In spring 1905 Einstein recognized Newton's Absolute Time to be the core problem. Well aware of the experimental facts, Einstein turned the problem into a basic principle or an axiom. The former problem became the foundation of a new theory. His analysis of the simultaneity of events detected the basic phenomena *time dilation*, *length contraction* and *desynchronisation*. In a couple of weeks he and his wife wrote down the seminal paper on STR published in June 1905. In the same way Einstein later resolved the problematic equivalence of inertial and gravitational mass. There is no reason for this experimental fact in classical physics, already Newton wondered about it. Einstein interprets the problem as a fundamental fact, and based on this new axiom he develops (on a long and laborious path) his General Theory of Relativity.

STR is based on the axiom

A1 The vacuum speed of light is a universal constant, independent of the state of movement of the source.

Classical physics knows four quantities conserved in all processes: mass, energy, momentum and electric charge. STR affects three of these conservation theorems; only conservation of total electric charge is unchanged.
In STR, conservation of mass and conservation of energy fall into one; 'conservation of dynamic mass' and 'conservation of total energy' are essentially the same. Conservation of total momentum is still valid, but 'STR momentum' differs slightly from momentum in Newton's theory.
Section 18 shows, without using any of the conservation theorems, how STR momentum necessarily has to be defined.
Section 19 shows how conservation of dynamic mass follows from conservation of momentum.
Section 4 derives the definitions of dynamic mass and STR momentum just by assuming the *existence* of a velocity-dependence of mass that makes conservation of mass and conservation of momentum true.

As a matter of fact all three of Newton's fundamental laws still hold in STR ! The first law deals with a special case of the second one and so needs no separate discussion. The second law says, in Newton's original formulation, $F = dp/dt$. This is still true, with some refinement in the definition of momentum ! And his third law ('actio = reactio') is a deep insight, in a realm high above the details of any specific theory.

If the equation $E = p \cdot c$ for some amount of radiation is taken as given then it is quite easy to derive the equivalence of mass and energy from conservation of momentum (sections 10 to 12) or from conservation of energy (section 13). We may use then a weaker version of axiom **A1** to derive the STR, the premise of the independence of light speed from movement of the source can be omitted. **A1** is equivalent to **A2 & A3**, if we define

A2 The vacuum speed of light emitted by a source at rest is an universal constant.

A3 Energy and momentum of electromagnetic radiation are linked by the equation $E = p \cdot c$.